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ON HALIN-LATTICES IN GRAPHS

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Halin [2] has shown that primitive sets with respect to a subset A of the vertex set of a connected graph G form a complete lattice (Halin-lattice). In this article special contractions are defined such that pairs (G, A) and these maps form a category HG and that a contravariant functor exists from HG to the category of complete lattices and lattice homomorphisms. Using this functor it is proved that the lattice homomorphism is a well-quasi ordering in the class of all Halin-lattices of rooted trees.

1. Introduction

In his paper Halin [2] has introduced the concept of a primitive set of vertices of a graph in the following sense: Holding a subset A of the vertex set of a connected graph G fixed as an 'origin', we can write $T \trianglelefteq T'$ if T separates A and T'. This relation is a quasi order on the power set of the vertex set of G. Now we can force $\trianglelefteq$ to be a partial order calling sets primitive if they represent certain equivalence classes. Then the set of all primitive sets endowed with the relation $\trianglelefteq$ is a complete lattice, called Halin-lattice (H-lattice).

In this paper we will define a category HG (Halin-graphs) whose objects are all pairs (G, A), G a connected graph and $A \subseteq V(G)$ a set of vertices and whose morphisms will be special kinds of contractions.

We are interested mainly in the case of trees, and for example in the case of 2-connected graphs it can happen that only a few morphisms or none exist.

The main theorem of this article is Theorem 4.7, which shows that there exists a contravariant functor $\mathfrak{F}$ from HG to the category V of all complete lattices and lattice homomorphisms such that *infima and suprema* are preserved, whereas Polat [5] and Sabidussi [6] construct categories and functors which only preserve infima for other kinds of primitive sets and lattices. The three types of primitivity are compared in [1], where it is also proved that each complete lattice can be described as an H-lattice.

By five examples we show that small variations of our definition of morphisms contradict the existence of a functor in our sense. In the last chapter of this article we will be concerned with rooted trees and prove in Theorem 5.4 that lattice homomorphism is a well-quasi ordering in the class of all H-lattices of rooted trees using our functor $\mathfrak{F}$ and a result of Nash-Williams [3, 4].

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2. Notations

For a graph G we denote the vertex set by $V(G)$, the edge set by $E(G)$ and the set of all neighbours of a vertex x in G by $N_G(x)$. If V is a set of vertices, $N_G(V)$ is the union of the sets $N_G(x) - V$, $x \in V$.

A *path* $W = (x_0, \ldots, x_n)$ is a graph with $V(W) = \{x_0, \ldots, x_n\}$ (all x_i distinct) and $E(W) = \{[x_{i-1}, x_i] \mid i = 1, \ldots, n\}$.

A *circuit* $C = (x_0, \ldots, x_n)$ is a graph which consists of the path $(x_0, \ldots, x_n)$ and the edge $[x_n, x_0]$.

An edge b in a connected graph, whose removal disconnects the graph such that each of the two components contains at least one edge, is called *bridge*.

An *endvertex* is a vertex x for which $|N_G(x)| = 1$.

A *restriction* X of a graph G is a subgraph of G in which $x, y \in V(X)$, $[x, y] \in E(G)$ always implies $[x, y] \in E(X)$.

For $B \subseteq V(G)$ let $G[B]$ be the restriction X of G with $V(X) = B$, and $G(B)$ the graph with $V(G(B)) = V(G) \cup \{i\}$, $i \notin V(G)$, and $E(G(B)) = E(G) \cup \{[i, b] \mid b \in B\}$.

Let $(G_i)_{i \in I}$ be a family of restrictions of the graph G, then $\bigcup_{i \in I} G_i$, resp. $\bigcap_{i \in I} G_i$, is the restriction X of G with $V(X) = \bigcup_{i \in I} V(G_i)$, resp. $\bigcap_{i \in I} V(G_i)$.

Let G, G' be connected graphs. G' is a *subdivision* of G ($G' = U(G)$) if there exists a set of edges $K \subseteq E(G)$ such that G' arises from G by replacing each $k \in K$ by a path $P(k)$.

A function $\phi: V(G) \to V(G')$, where G and G' are connected graphs, is called *contraction*, if ϕ satisfies the following conditions:

(i) If $v \in V(G')$, then $G[\phi^{-1}(v)]$ is connected.

(ii) If $[v, w] \in E(G)$, then either $\phi(v) = \phi(w)$ or $\phi(v)$ is adjacent to $\phi(w)$.

(iii) If $[x, y] \in E(G')$, then there is an edge $[v, w] \in E(G)$ with $\phi(v) = x$ and $\phi(w) = y$.

Hence each contraction is surjective.

A *tree* is a connected graph B with no circuits and a *rooted tree* is a tree with a distinguished vertex w. $V(B)$ can be ordered by: $v \leqslant_w v'$ iff the only existing path from w to v' contains v. Let (B, w), (B', w') be rooted trees:

(i) $(B, w) \subseteq (B', w')$ means that B is a subtree of B' and $w = \min_{w'} V(B)$.

(ii) $(B, w) <_U (B', w')$ means that there is a rooted tree $(B'', w'') \subseteq (B', w')$, with (B'', w'') is isomorphic to a subdivision $(U(B), w)$.

A *complete lattice* is a set V with a partial order $\leqslant$ such that $\sup_{i \in I} v_i$ and $\inf_{i \in I} v_i$ exist for each family $(v_i)_{i \in I}$ of elements of V. Let $(V, \leqslant)$ and $(V', \leqslant')$ be complete lattices and $f: V \to V'$ a map for which holds:

$$f\left(\sup_{i \in I} v_i\right) = \sup_{i \in I}{}' f(v_i), \qquad f\left(\inf_{i \in I} v_i\right) = \inf_{i \in I}{}' f(v_i)$$

for every family $(v_i)_{i \in I}$ in V, then f is called a *lattice homomorphism*.

Let $(M, \trianglelefteq)$ be a class with a quasi-ordering, which means that $\trianglelefteq$ is reflexive and transitive, then we call $\trianglelefteq$ a *well-quasi ordering* on M if for each infinite sequence $m_1, m_2, \ldots$ in M there exist subscripts $i<j$ with $m_i \trianglelefteq m_j$.

3. *H*-primitive sets (see Halin [2])

Let G be a connected graph and $A \subseteq V(G)$ a fixed set of vertices. Let $T \subseteq V(G)$ and let $W(A, T)$ be the set of all paths $W=(x_0, \ldots, x_n)$ such that $x_0 \in A$ and $x_0, \ldots, x_{n-1} \notin T$. The restriction of G to $\bigcup W(A, T)$ is called *connection graph from A to T* ('*Verbindungsgraph*') and is denoted by $G(A \to T)$.

If $T' \subseteq T''$, then $G(A \to T'') \subseteq G(A \to T')$.

Two sets T', T'' are A-equivalent, if $G(A \to T') = G(A \to T'')$.

Then each equivalence-class contains one smallest set of vertices, relative to inclusion, the set $\beta_A(T)$ of those vertices which are adjacent to at least one vertex of $G - G(A \to T)$.

In general $\beta_A(T) \subseteq T$.

If $\beta_A(T) = T$, then T is called an *H-primitive set* (relative to (G, A)).

Let T', T'' be two H-primitive sets and $G(A \to T') \subseteq G(A \to T'')$ then each path from A to T'' meets T', therefore T' separates T'' and A [2, (1.5)].

Now we can define a partial order '$\trianglelefteq$' on the set $H(G, A)$ of all H-primitive sets: $T' \trianglelefteq T''$ if and only if $G(A \to T') \subseteq G(A \to T'')$.

Then $H(G, A)$ forms a complete lattice with respect to $\trianglelefteq$, called *H-lattice* [2, Satz 1].

Let T_i, $i \in I$, be H-primitive sets. Then

$$\inf_{i \in I} T_i = \beta_A\Bigl(\bigcup_{i \in I} T_i\Bigr) = R \quad \text{and} \quad G(A \to R) = G\Bigl(A \to \bigcup_{i \in I} T_i\Bigr) \subseteq \bigcap_{i \in I} G(A \to T_i);$$

$$\sup_{i \in I} T_i = T \quad \text{and} \quad G(A \to T) = \bigcup_{i \in I} G(A \to T_i),$$

here $v \in T$ if and only if v is adjacent to at least one vertex of $G - \bigcup_{i \in I} G(A \to T_i)$.

4. The category HG

We define a category HG (*Halin-Graphs*) whose objects are all pairs (G, A), G a connected graph and $A \subseteq V(G)$ a set of vertices, and for which there exists a functor $\mathfrak{F}$ from HG to the category V of all complete lattices; that $\mathfrak{F}$(HG) includes all complete lattices is proved in [1].

Let Mor HG be the class of all morphisms in HG and $\mathrm{Mor}_{\mathrm{HG}}((G, A), (G', A'))$ the class of all morphisms between (G, A) and (G', A') in HG.

4.1. Definition. (a) Let G be a connected graph and $A \subseteq V(G)$. Then (G, A) is an object of the class HG.

(b) Let G', G'' be connected graphs, $A' \subseteq V(G')$, $A'' \subseteq V(G'')$. Then let $\alpha_1, \alpha_2, \alpha_3$ be contractions from $V(G')$ to $V(G'')$ which satisfy:

α_1: (i) $\alpha_1^{-1}(A'') = A'$.
(ii) If $\alpha_1(a) = \alpha_1(b)$, $[a, b] \in E(G')$, then $N_{G'}(a) - \{b\} = N_{G'}(b) - \{a\}$.

α_2: (i) $\alpha_2(A') = A''$.
(ii) If $\alpha_2(a) = \alpha_2(b)$, $[a, b] \in E(G')$, then $[a, b]$ must be a bridge in $G'(A')$.

α_3: (i) $\alpha_3(A') = A''$.
(ii) If $\alpha_3(a) = \alpha_3(b)$, $[a, b] \in E(G')$, then $[a, b]$ must contain an endvertex, for example a, and for each circuit C in $G'(A')$ must hold: $b \notin V(C)$.

The contractions α_i, $i = 1, 2, 3$, are called *elementary morphisms*.

(c) ϕ is a morphism of Mor HG if ϕ is the composition of a finite number of elementary morphisms (therefore ϕ is a contraction).

4.2. Theorem. HG *is a category.*

This is an immediate consequence of 4.1.

The following are examples of elementary morphisms:

(i) Let G, G' be connected graphs and $(G_i)_{i \in I}$ a disjoint family of connected subgraphs of G. If there exists a contraction $\phi: V(G) \to V(G')$, each G_i is a complete graph, $N_G(x) - V(G_i) = N_i$ holds for all $i \in I$ and $x \in V(G_i)$, and $\phi(G_i)$ is a single vertex in G', then $\phi: (G, \phi^{-1}(A)) \to (G', A)$ is an elementary morphism of type α_1 for each $A \subseteq V(G')$.

(ii) Let G, G' be connected graphs and $U(G') = G$. Let $A \subseteq V(G')$ and $E' \subseteq E(G')$ be the set of edges which have been subdivided. If no $k \in E'$ belongs to a circuit in $G(A)$, then there exists $\phi: (G, A) \to (G', A)$, an elementary morphism of type α_2.

If $\phi: (G, A) \to (G', A')$ is a morphism then ϕ consists of elementary morphisms. Therefore let $\phi = \gamma_n \cdots \gamma_1$, $\gamma_i: (G_i, A_i) \to (G_{i+1}, A_{i+1})$ be elementary morphisms, with $(G_1, A_1) = (G, A)$, $(G_n, A_n) = (G', A')$. In the following we denote $\bar{\gamma}_j := \gamma_j \cdots \gamma_1$, $\gamma_{j+1}^* := \gamma_{j+1}^{-1} \cdots \gamma_n^{-1}$.

Now we will prove a technical lemma which is essential for the following part of this section.

4.3. Lemma. *Let be $\phi \in \mathrm{Mor}_{\mathrm{HG}}((G, A), (G', A'))$, $[x_1, x_2]$ an edge of G. If j is the least subscript with $\bar{\gamma}_j(x_1) = \bar{\gamma}_j(x_2)$, if γ_j is an α_1-morphism and G_{j+1} contains more than one vertex, then each elementary morphism γ_l, $l < j$, which contracts an edge in $\bar{\gamma}_j^{-1}(\bar{\gamma}_j(x_i))$, $i \in \{1, 2\}$, is an α_1-morphism.*

Proof. Since $|V(G_{j+1})| \geq 2$, there exists a vertex $y \in V(G)$ with $[\bar{\gamma}_j(y), \bar{\gamma}_j(x_1)] \in E(G_{j+1})$. Therefore $z \in \gamma_j^{-1}(\bar{\gamma}_j(y))$ exists, which is adjacent to a vertex $u \in \gamma_j^{-1}(\bar{\gamma}_j(x_1))$. ($\phi$ and each γ_i are contractions.)

Now each vertex $x \in \gamma_j^{-1}(\bar{\gamma}_j(x_1))$ is adjacent to z, because γ_j is an α_1-map. Hence x belongs to a circuit in $G_j(A_j)$.

Let be $l<j$ the greatest subscript such that $|\gamma_{j-1}^{-1} \cdots \gamma_l^{-1}(x)| \geq 2$ for such a vertex x. Thus γ_l cannot be of type α_2 or α_3, because there is a circuit.

Therefore γ_l is an α_1-map and by repeating the argument one sees that the same holds for each γ_i which contracts an edge in $\bar{\gamma}_j^{-1}(\bar{\gamma}_j(x_i))$.

4.4. Lemma. *Let $\phi \in \mathrm{Mor}_{\mathrm{HG}}((G, A), (G', A'))$ and let $T \subseteq V(G')$ be an H-primitive set relative to (G', A').*

If $x \in \phi^{-1}(T)$ and $x \in V(G(A \to \phi^{-1}(T)))$, then $x \in \beta_A \phi^{-1}(T)$.

Proof. Since $\phi(x) \in T$ there exists a vertex $b \in V(G')$, $b \notin V(G'(A' \to T))$, which is adjacent to $\phi(x)$. Hence a vertex $z \in \phi^{-1}(\phi(x))$ and a vertex $b' \in \phi^{-1}(b)$ exist with $[b', z] \in E(G)$.

If $[x, b'] \in E(G)$, then $x \in \beta_A \phi^{-1}(T)$.

Let us assume now that $[x, b'] \notin E(G)$. Thus $x \neq z$ and so a shortest path $P = (x_1, \ldots, x_r)$ exists in $\phi^{-1}(\phi(x))$, which links x to z and contains at least one edge. No edge of P can be contracted by an α_3-morphism if not $A = \{x\}$ and therefore $\{x\} = \beta_A \phi^{-1}(T)$.

Also no α_1-morphism γ_l can contract an edge $k = [x_i, x_{i+1}]$ of P, because out of 4.3 we get a contradiction to the fact that P is a shortest path or that $[x, b'] \notin E(G)$. Therefore each edge of P must be a bridge and so $x \in \beta_A \phi^{-1}(T)$ holds.

Corollary. *Let $\phi \in \mathrm{Mor}_{\mathrm{HG}}((G, A), (G', A'))$ and suppose $T \subseteq V(G')$ is a primitive set relative to (G', A'). Let W be a path from A to x with $\beta_A \phi^{-1}(T) \cap V(W) \subseteq \{x\}$, then it follows that $\phi^{-1}(T) \cap V(W) \subseteq \{x\}$.*

4.5. Lemma. *Let $\phi \in \mathrm{Mor}_{\mathrm{HG}}((G, A), (G', A'))$ and $T \subseteq V(G')$ a primitive set relative to (G', A'), then for all $y \in T$ at least one of the following assertions holds:*

(i) *$\{y\}$ is a primitive set relative to (G', A').*

(ii) *$\phi^{-1}(y) \subseteq V(G(A \to \phi^{-1}(T)))$.*

Proof. Given $\phi = \gamma_n \cdots \gamma_1$ let γ_j be the map with the first index such that $|\bar{\gamma}_j(\phi^{-1}(y))| = 1$. If $|\phi^{-1}(y))| = 1$, (ii) holds.

Therefore assume $|\phi^{-1}(y)| \geq 2$. Now there exists $b \in V(G')$, $[b, y] \in E(G')$, $b \notin V(G'(A' \to T))$, because $y \in T$. Hence an edge k between $\gamma_j^*(y)$ and $\gamma_j^*(b)$ exists.

(a) If γ_j is of type α_3 or of type α_2, then (i) holds.

Proof. Let γ_j be of type α_3, then no vertex in $\gamma_j^*(y)$ can belong to a circuit in $G_j(A_j)$. So (i) holds because γ_i $(i = 1, \ldots, n)$ are contractions and $[b, y]$ exists.

Now let γ_j be of type α_2, then each edge in $\gamma_j^*(y)$ must be a bridge in $G_j(A_j)$.

Therefore at least one path W beginning at A_j exists which goes through $\gamma_j^*(y)$. On this path there is an edge $k' \in \gamma_j^*(y)$ which will be contracted by γ_j. k' is also a bridge and so there exist $c \notin \gamma_j^*(y)$, $y' \in \gamma_j^*(y)$ so that $[c, y']$ and k' are incident with y'. Hence $\gamma_j^*(y)$ separates A_{j+1} and $\gamma_j(c)$. So (i) holds.

(b) If γ_j belongs to type α_1, then (ii) holds.

Proof. (b.1) $y \notin A_2$, then $\phi^{-1}(y) \cap A_1 = \emptyset$. Thus there exists a vertex $x' \in \gamma_j^*(y)$ which is adjacent to a vertex $r' \in V(G_j(A_j \to \gamma_j^*(T))) - \gamma_j^*(T)$, $r' \in \gamma_j^*(r)$ and a vertex $x'' \in \gamma_j^*(y)$ which is adjacent to a vertex $b' \in \gamma_j^*(b)$. Therefore each $x \in \gamma_j^*(y)$ is adjacent to r' and b' because γ_j has type α_1. From 4.3 we know that each γ_l which contracts edges in $\phi^{-1}(y)$ is an α_1-map. Therefore (ii) holds.

(b.2) $y \in A_2$, then $\gamma_{j+1}^*(y) \in A_{j+1}$. Thus $\gamma_j^*(y) \subseteq A_j$ because γ_j is an α_1-map. With 4.3 we get $\phi^{-1}(y) \subseteq A$. Hence (ii) holds.

4.6. Definition. Let (G, A), (G', A') be objects of HG and ϕ a morphism of $\mathrm{Mor}_{\mathrm{HG}}((G, A), (G', A'))$, then we call the complete lattice of the primitive sets relative to (G, A), resp. (G', A'), $\mathfrak{F}(G, A)$, resp. $\mathfrak{F}(G', A')$.

Also we define $\mathfrak{F}(\phi)$ by $\beta_A \phi^{-1} : \mathfrak{F}(G', A') \to \mathfrak{F}(G, A)$.

Now we can prove Theorem 4.7, the main theorem of this section.

4.7. Theorem. *$\mathfrak{F}$ is a contravariant functor from the category* HG *to the category* V *of all complete lattices and lattice homomorphisms.*

Proof. (a) Let $(G, A) \in \mathrm{HG}$. Then $\mathfrak{F}(G, A) \in \mathrm{V}$.

(b) Let $\phi \in \mathrm{Mor}_{\mathrm{HG}}((G, A), (G', A'))$. Then $\mathfrak{F}(\phi)$ has to be an element of $\mathrm{Mor}_{\mathrm{V}}(\mathfrak{F}(G', A'), \mathfrak{F}(G, A))$. Hence we have to show that $\beta_A \phi^{-1}$ preserves arbitrary infima and suprema.

Let $(T_i)_{i \in I}$ be a family of primitive sets of $\mathfrak{F}(G', A')$.

(b.1) Let $S := \sup_{i \in I} \beta_A \phi^{-1}(T_i)$ and $T := \sup_{i \in I} T_i$. Then we must show that $S = \beta_A \phi^{-1}(T)$. This is equivalent to $G(A \to S) = G(A \to \phi^{-1}(T))$. We know that $G(A \to S) = \bigcup_{i \in I} G(A \to \phi^{-1}(T_i))$ holds. So we only have to prove that $G(A \to \phi^{-1}(T)) \subseteq \bigcup_{i \in I} G(A \to \phi^{-1}(T_i))$, because the inverse inclusion is an immediate consequence of $T_i \trianglelefteq T$ for each $i \in I$ and the fact that ϕ is a contraction.

Let $x \in V(G(A \to \phi^{-1}(T)))$. Then there is a path W from A to x with $V(W) \cap \phi^{-1}(T) \subseteq \{x\}$. Thus $\phi(V(W))$ is the set of vertices of a path W' from A' to $\phi(x)$ with $V(W') \cap T \subseteq \{\phi(x)\}$. So $\phi(x)$ is a vertex of $G'(A' \to T) = \bigcup_{i \in I} G'(A' \to T_i)$. Hence there is a $j \in I$ with $\phi(x) \in V(G'(A' \to T_j))$ which implies the existence of a path W^* from A' to $\phi(x)$ with $V(W^*) \cap T_j \subseteq \{\phi(x)\}$. Thus $\phi^{-1}(V(W^*))$ contains the vertex set of a path W^{**} from A to $x \in \phi^{-1}(\phi(x))$ with $V(W^{**}) \cap \phi^{-1}(T_j) \subseteq \phi^{-1}(\phi(x))$. Supposing that $V(W^{**}) \cap \phi^{-1}(T_j) = \emptyset$, then (b.1) holds.

Assume therefore that $V(W^{**}) \cap \phi^{-1}(T_j) \neq \emptyset$. Then $\phi(x) \in T_j$. By 4.5 either (b.1) holds or $\{\phi(x)\}$ has to be an element of $\mathfrak{F}(G', A')$. So let $\{\phi(x)\} \in \mathfrak{F}(G', A')$.

Suppose that $\phi(x) \notin T$. Then there exists a neighbour y of $\phi(x)$ such that y belongs to a component of $G'(A') - \phi(x)$ which is separated from A'. Therefore a primitive set T_k exists with $\phi(x) \in V(G'(A' \to T_k)) - T_k$. So (b.1) holds.

If $\phi(x) \in T$, then $x \in A$, whence (b.1) holds, or x has to be adjacent to a vertex z with $\phi(z) \in V(G'(A' \to T)) - T$. Let $\phi(z) \notin V(G'(A' \to T_j)) - T_j$, otherwise we have (b.1). Now a circuit C in $G'(A')$ with $\phi(x) \in V(C)$ exists. Let $\phi = \gamma_n \cdots \gamma_1$ and γ_j the elementary map with the least index such that $|\bar{\gamma}_j(\phi^{-1}(\phi(x)))| = 1$. Then γ_j is an α_2-map if $\phi^{-1}(\phi(x)) \nsubseteq V(G(A \to \phi^{-1}(T)))$ (see Proof of 4.5), otherwise (b.1) holds. Hence it follows that $G[\gamma_j^*(\phi(x))]$ is the union of a collection of paths with the only common vertex $\overline{\gamma_{j-1}}(x)$, i.e., a star, and only $\overline{\gamma_{j-1}}(x)$ is adjacent to vertices v_i with $\phi(v_i) \in V(G'(A' \to T)) - T$.

Let k be the greatest index $<j$ for which γ_k is an α_1-map and γ_k has contracted an edge in $\gamma_k^*(x)$, then all maps γ_l, $k < l \leq j$, which contract edges in $\gamma_l^*(x)$, have to be α_2-maps, and $\overline{\gamma_{l-1}}(x)$ is the only vertex which is adjacent to vertices w_i with $\phi(w_i) \in V(G'(A' \to T)) - T$. If k does not exist, then (b.1) holds because $[x, z] \in E(G)$. Hence $\gamma_k^*(x) \subseteq V(G_k(A_k \to \gamma_k^*(T_j)))$. Thus each map γ_m, $m < k$, which contracts edges in $\gamma_m^*(x)$ to a vertex t, has to be an α_1-map (see 4.3). Therefore (b.1) holds.

(b.2) Secondly we have to show that

$$R := \inf_{i \in I} \beta_A \phi^{-1}(T_i) = \beta_A \phi^{-1}\left(\inf_{i \in I} T_i\right).$$

We know:

$$\beta_{A'} \bigcup_{i \in I} T_i = \inf_{i \in I} T_i, \qquad R = \beta_A \bigcup_{i \in I} \beta_A \phi^{-1}(T_i).$$

Thus we have to prove:

$$G\left(A \to \phi^{-1}\left(\beta_{A'} \bigcup_{i \in I} T_i\right)\right) = G\left(A \to \bigcup_{i \in I} \beta_A \phi^{-1}(T_i)\right).$$

(b.2.1) Let $x \in V(G(A \to \phi^{-1}(\beta_{A'} \bigcup_{i \in I} T_i)))$. Then there exists a path W from A to x with $V(W) \cap \phi^{-1}(\beta_{A'} \bigcup_{i \in I} T_i) \subseteq \{x\}$. Thus $\phi(V(W))$ is the vertex-set of a path W' from A' to $\phi(x)$ with

$$V(W') \cap \beta_{A'} \bigcup_{i \in I} T_i \subseteq \{\phi(x)\}.$$

Hence it follows that $V(W') \cap \bigcup_{i \in I} T_i \subseteq \{\phi(x)\}$ because each $y \in T_j$ with $j \in I$, $y \in V(W')$ is an element of a primitive set which separates b from A', $b \notin G'(A' \to \bigcup_{i \in I} T_i)$, $[b, y] \in E(G')$, so $y \in \beta_{A'} \bigcup_{i \in I} T_i$. Therefore

$$V(W) \cap \phi^{-1}\left(\bigcup_{i \in I} T_i\right) \subseteq \{x\}$$

holds because $y \in V(W) \cap \phi^{-1}(\bigcup_{i \in I} T_i)$ implies $\phi(y) = \phi(x)$ and so $y \in \phi^{-1}(\beta_{A'}$

$\bigcup_{i\in I} T_i)$. Since $\phi^{-1}(\bigcup_{i\in I} T_i)=\bigcup_{i\in I}\phi^{-1}(T_i)$ we have $V(W)\cap\bigcup_{i\in I}\phi^{-1}(T_i)\subseteq\{x\}$ and also

$$V(W)\cap\bigcup_{i\in I}\beta_A\phi^{-1}(T_i)\subseteq\{x\}.$$

(b.2.2) Let $x\in V(G(A\to\bigcup_{i\in I}\beta_A\phi^{-1}(T_i)))$. Then there exists a path W from A to x with $V(W)\cap\bigcup_{i\in I}\beta_A\phi^{-1}(T_i)\subseteq\{x\}$. Thus $V(W)\cap\beta_A\phi^{-1}(T_i)\subseteq\{x\}$ holds for each $i\in I$. From the corollary to 4.4 we get: $V(W)\cap\phi^{-1}(T_i)\subseteq\{x\}$ holds for each $i\in I$. This means:

$$V(W)\cap\bigcup_{i\in I}\phi^{-1}(T_i)=V(W)\cap\phi^{-1}\Big(\bigcup_{i\in I}T_i\Big)\subseteq\{x\}.$$

Hence $V(W)\cap\phi^{-1}(\beta_{A'}\bigcup_{i\in I}T_i)\subseteq\{x\}$.

(c) It is clear that $\mathfrak{F}$ maps the identity on (G,A) to the identity on $\mathfrak{F}(G,A)$.

(d) Let $\phi:(G,A)\to(G',A')$, $\psi:(G',A')\to(G'',A'')$ be morphisms. We have to show that $\mathfrak{F}(\psi\cdot\phi)=\mathfrak{F}(\phi)\cdot\mathfrak{F}(\psi)$. This means that

$$\beta_A\phi^{-1}\psi^{-1}(K)=\beta_A\phi^{-1}\beta_{A'}\psi^{-1}(K)$$

holds for an arbitrary $K\in\mathfrak{F}(G'',A'')$. Therefore we only have to prove that

$$G(A\to\phi^{-1}\psi^{-1}(K))\supseteq G(A\to\phi^{-1}\beta_{A'}\psi^{-1}(K))$$

because the opposite inclusion is immediate.

Let $x\in V(G(A\to\phi^{-1}\beta_{A'}\psi^{-1}(K)))$. Then there is a path W from A to x with $V(W)\cap\phi^{-1}\beta_{A'}\psi^{-1}(K)\subseteq\{x\}$. Thus $\phi(V(W))$ is the vertex-set of a path W' from A' to $\phi(x)$ with $V(W')\cap\beta_{A'}\psi^{-1}(K)\subseteq\{\phi(x)\}$. From the corollary to 4.4 we get: $V(W')\cap\psi^{-1}(K)\subseteq\{\phi(x)\}$ and so $V(W)\cap\phi^{-1}\psi^{-1}(K)\subseteq\phi^{-1}(\phi(x))$. Supposing there is a vertex $y\in\phi^{-1}\psi^{-1}(K)$, $y\in V(W)$, then $\phi(y)=\phi(x)$ and $\phi(x)\in\psi^{-1}(K)$. From 4.4 we obtain $\phi(x)\in\beta_{A'}\psi^{-1}(K)$ and it follows that $x=y$.

This completes the proof of Theorem 4.7.

4.8. Proposition. *If $\phi\in\mathrm{Mor}_{HG}((G,A),(G',A'))$, then $\mathfrak{F}(\phi)$ is a monomorphism.*

Proof. Suppose that $T,T'\in\mathfrak{F}(G',A')$ satisfy $\beta_A\phi^{-1}(T)=\beta_A\phi^{-1}(T')$, then we have to show that $T'=T$. This follows at once if $\phi(G(A\to\phi^{-1}(T)))=G'(A'\to T)$ holds for each $T\in\mathfrak{F}(G',A')$. Let W be a path from A to $x\in V(G)$. Then it is obvious that $\phi(W)$ is a path from A' to $\phi(x)$ because $\phi(A)=A'$ and ϕ is a contraction. Therefore $\phi(G(A\to\phi^{-1}(T)))\subseteq G'(A'\to T)$.

If W is a path from A' to y, then $\phi^{-1}(W)$ contains a path from A to a vertex $z\in\phi^{-1}(y)$. Thus $\phi^{-1}(G'(A'\to T))\subseteq G(A\to\phi^{-1}(T))$. Hence $\phi(G(A\to\phi^{-1}(T)))=G(A'\to T)$. (See also [5, (3.7)] and [6, (5.6)].)

Now we will look at five examples of maps ϕ which are not morphisms of our

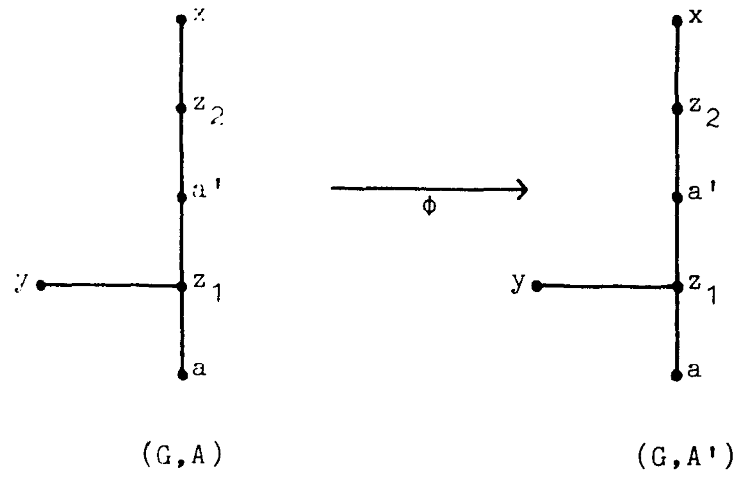

Fig. 1.

category, but show that our conditions for morphisms are sharp if we contract at most one edge.

Example A. See Fig. 1. Let ϕ be the identity on G and $A=\{a\}$, $A'=\{a, a'\}$. Let be $T_1=\{z_1\}$, $T_2=\{z_2\}$, then $\sup(T_1, T_2)=\emptyset$ in (G, A') and $\sup(T_1, T_2)=\{z_2\}$ in (G, A). Thus $\beta_A\phi^{-1}$ does not preserve suprema. The same holds for the inverse map $\beta_{A'}\phi$. Therefore $\phi(A)=A'$ is a necessary condition for morphisms.

Example B. See Fig. 2. Let ϕ be the identity on $G-\{a_1, y\}=G'-\{a_1'\}$ and $\phi(a_1)=\phi(y)=a_1'$. Let $A=\{a_1, a_2, a_3\}$, $A'=\{a_1', a_2, a_3\}$. Then ϕ fulfills the conditions of an α_1-map; only the term $A=\phi^{-1}(A')$ is weakened to $\phi(A)=A'$. Now $\beta_A\phi^{-1}$ does not preserve suprema because with $T_1=\{a_1', a_2, z_2, z_3\}$, $T_2=\{a_1', a_3, z_1, z_4\}$ we get $\sup(T_1, T_2)=\{a_1'\}$ and $\sup(\beta_A\phi^{-1}(T_1), \beta_A\phi^{-1}(T_2))=\{a_1, z_2, z_4\}$.

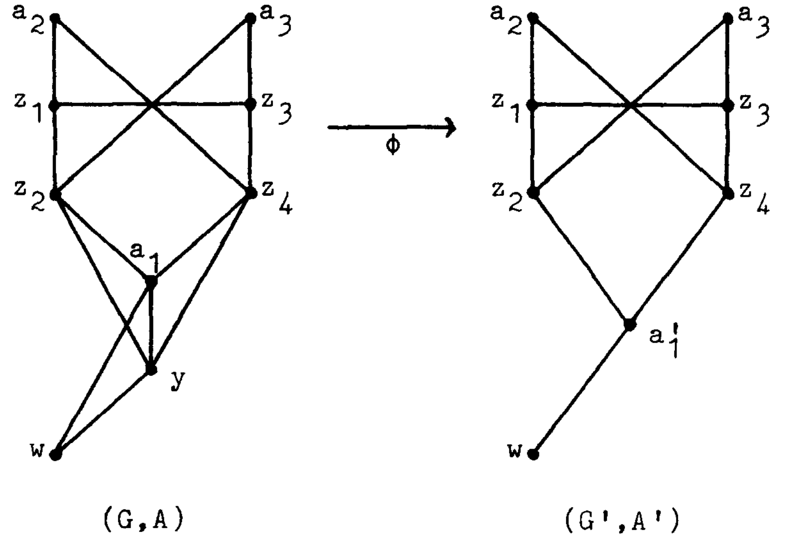

Fig. 2.

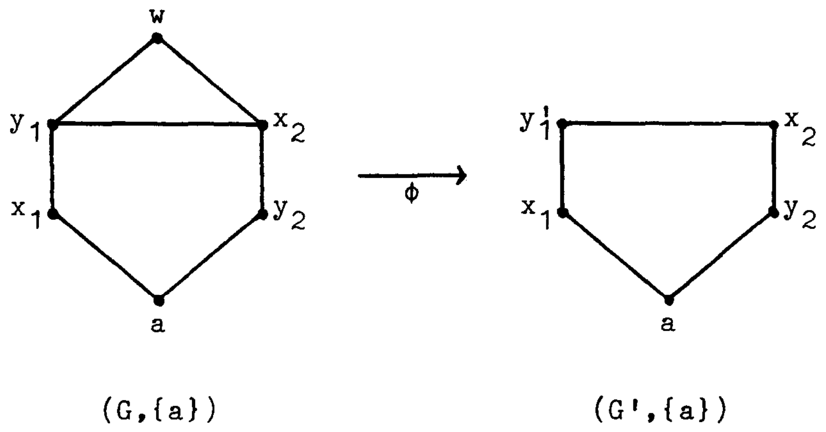

Fig. 3.

Example C. See Fig. 3. Let ϕ be the identity on $G-\{y_1, w\}$ and $\phi(y_1)=\phi(w)=y_1'$, then ϕ is nearly an α_1-map except that $N_G(w)\neq N_G(y_1)$. Let $T_1=\{x_1, x_2\}$, $T_2=\{y_1', y_2\}$. Then $\sup(T_1, T_2)=\emptyset$ in $(G', \{a\})$ and $\sup(\beta_A\phi^{-1}(T_1), \beta_A\phi^{-1}(T_2))=\{y_1, x_2\}$.

Example D. Let G', ϕ be defined as in Example C and G changed into $G^*=G-\{[y_1, x_2]\}$, then there is only one circuit through $[y_1, w]$ and $[y_1, w]$ is not a bridge. Now ϕ is nearly an α_2-map,. but as we have seen before, $\beta_A\phi^{-1}$ does not preserve all suprema.

Example E. Let G', ϕ be defined as in Example C and G changed into $G^{**}=G-\{[w, x_2]\}$, then ϕ is nearly an α_3-map, but there exists a circuit through y_1. As we have seen before $\beta_A\phi^{-1}$ does not preserve all suprema.

5. Rooted trees and morphisms

5.1. Lemma. *Let B, B' be two trees with $B=U(B')$ and w a root of B', then there exists a morphism $\phi:(B, w)\to(B', w)$.*

Proof. Rooted trees are objects of HG. Now $B(\{w\})$ does not contain any circuit and therefore each contracted edge is a bridge or contains an endvertex. Thus $\phi=\gamma_1\gamma_2$ exists with γ_1 an α_3-map and γ_2 an α_2-map.

5.2. Lemma. *Let (B, w), (B', w') be rooted trees with $(B', w')\subseteq(B, w)$, then there exists a morphism $\phi:(B, w)\to(B', w')$.*

Proof. $(B', w')\subseteq(B, w)$ means that B' is a subtree of B and $w'=\min_w V(B')$. Put

$$V^*=V(B(\{w\}\to\{w'\}))-\{w'\}.$$

Then $V^*\cap V(B')=\emptyset$. Now let $N(b)=N_B(b)-N_{B'}(b)$, $b\in V(B')$. Hence it follows that each b' which is an element of a path beginning at w and follows a vertex $c \in N(b)$, is not an element of $V(B')$. Therefore c separates b' and B'. For each $c \in N_B(b)$ let $V_c = \{x \in V(B) \mid$ the only path from w to x contains $[b, c]\}$, and $D_b = \bigcup_{c \in N_B(b)} V_c$. Thus $D_b \cap V(B') = \emptyset$ holds and the sets $(D_b)_{b \in V(B')}$ are disjoint. Hence

$$V(B) = V^* \dot{\cup} \bigcup_{b \in V(B')} D_b \dot{\cup} V(B').$$

Now we define ϕ. Consider $x \in V(B)$:

(i) If $x \in V(B')$, then $\phi(x) = x$.

(ii) If $x \in V^*$, then $\phi(x) = w'$.

(iii) If $x \in D_b$ with $b \in V(B')$, then $\phi(x) = b$.

So $\phi = \gamma_1 \gamma_2$ with γ_2 an α_2-map and γ_1 an α_3-map.

5.3. Theorem. *Let (B, w), (B', w') be rooted trees with $(B', w') <_U (B, w)$. Then there exists $\phi \in \mathrm{Mor}_{HG}((B, w), (B', w'))$ with $\mathfrak{F}(\phi) \in \mathrm{Mor}_V(\mathfrak{F}(B', w'), \mathfrak{F}(B, w))$.*

Proof. $(B', w') <_U (B, w)$ means: there is a rooted tree $(C, v) \subseteq (B, w)$ with (C, v) isomorphic to $(U(B'), w')$. Hence $\phi_1 \in \mathrm{Mor}_{HG}((B, w), (C, v))$ exists by 5.2, $\phi_2 \in \mathrm{Mor}_{HG}((B, w), (U(B'), w'))$ by the fact that each isomorphism induces a morphism, and $\phi_3 \in \mathrm{Mor}_{HG}((U(B'), w'), (B', w'))$ by 5.1. $\phi = \phi_3 \phi_2 \phi_1$ is an element of $\mathrm{Mor}_{HG}((B, w), (B', w'))$ by 4.2 and therefore $\mathfrak{F}(\phi) \in \mathrm{Mor}_V(\mathfrak{F}(B', w'), \mathfrak{F}(B, w))$ holds by 4.7.

5.4. Theorem. *Lattice homomorphism is a well-quasi-ordering in the class of all H-lattices of rooted trees.*

Proof. Let $V_1, V_2, \ldots$. be an infinite sequence of H-lattices of rooted trees. Then there has to exist at least one rooted tree (B_i, w_i) with $V_i = \mathfrak{F}(B_i, w_i)$ for each $i \in \mathbb{N}$. Now we choose for each $i \in \mathbb{N}$ a rooted tree (B_i, w_i) for which $V_i = \mathfrak{F}(B_i, w_i)$, thus obtaining a sequence (B_1, w_1), $(B_2, w_2), \ldots$ of rooted trees. By the theorem of Nash-Williams [3, 4] we know that $<_U$ is a well-quasi-ordering in the class of all rooted trees. Therefore there are two rooted trees (B_j, w_j), (B_k, w_k) with $j < k$ and $(B_j, w_j) <_U (B_k, w_k)$. Hence by 5.3, there is a $\phi \in \mathrm{Mor}_{HG}((B_k, w_k), (B_j, w_j))$ such that $\mathfrak{F}(\phi) \in \mathrm{Mor}_V(\mathfrak{F}(B_j, w_j), \mathfrak{F}(B_k, w_k))$. So $\mathfrak{F}(\phi) \in \mathrm{Mor}_V(V_j, V_k)$.

References

[1] M. Hager, On primitive sets in graphs, Europ. J. Combinatorics 3 (1982) 29–34.

[2] R. Halin, Über trennende Eckenmengen in Graphen und den Mengerschen Satz, Math. Annalen 157 (1964) 34–41.

[3] C. St. J. A. Nash-Williams, On well-quasi ordering finite trees, Proc. Cambridge Phil. Soc. 59 (1963) 833–835.
[4] C. St. J. A. Nash-Williams, On well-quasi ordering infinite trees, Proc. Cambridge Phil. Soc. 61 (1965) 697–720.
[5] N. Polat, Treillis de séparation des graphs, Canad. J. Math. 28 (1976) 725–752.
[6] G. Sabidussi, Weak separation lattices of graphs, Canad. J. Math. 28 (1976) 691–724.